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Magnetic transport in a straight parabolic channel

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Abstract

We study a charged two-dimensional particle confined to a straight parabolic-potential channel and exposed to a homogeneous magnetic field under the influence of a potential perturbation W. If W is bounded and periodic along the channel, a perturbative argument yields the absolute continuity of the bottom of the spectrum. We show that it can have any finite number of open gaps provided that the confining potential is sufficiently strong. However, if W depends on the periodic variable only, we prove by the Thomas argument that the whole spectrum is absolutely continuous, irrespective of the size of the perturbation. On the other hand, if W is small and satisfies a weak localization condition in the the longitudinal direction, we prove by the Mourre method that a part of the absolutely continuous spectrum persists.

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1. Introduction

The problem of magnetic transport goes back to the early 1980s [Ha, MS] when it was found that the transport can be achieved in a system with a homogeneous magnetic field if boundaries are present. These so-called edge currents found numerous applications in solid-state physics. Recently it has been shown that that such a type of transport exists even when the boundary is replaced by a periodic array of point obstacles [U, EJK]; in this case the propagation along the array is a purely quantum effect.

On the other hand, it was also recognized that a suitable translationally symmetric variation of the magnetic field itself can induce transport. A simple and transparent example of such a variation is provided by a step of the magnetic field intensity. As with the conventional edge states, the propagation here can also be understood at the classical level, since the cyclotronic radius at both sides of the step is different, see [CFKS, section 6.5]. Similarly transport can exist in the case when the magnetic field has the same asymptotics in both directions perpendicular to the field variation [Iw, MP, EK].

Naturally, it is of both theoretical and practical interest to understand how such a magnetic transport is influenced by various perturbations. Recently several studies treated the problem of edge-current stability with respect to a sufficiently weak 'random' perturbation (i.e. a deterministic bounded the potential of an arbitrary shape). In these works the particle was supposed to be confined in a semi-infinite region by either a smooth potential wall which vanishes in one half-plane and rapidly increases in the other [MMP], or by a Dirichlet boundary [BP,FGW]. The proofs were based on commutator methods. In [MMP] it was shown, using a version of the virial theorem, that in certain parts of the spectrum the Hamiltonian of the particle cannot have any eigenstates, so that the spectrum there is purely continuous. In [BP, FGW] the Mourre theory of positive commutators was used to prove that for energy intervals away from the Landau levels the spectrum remains purely absolutely continuous, i.e. that the transport survives in the presence of an impurity potential. Moreover, the argument of [FGW] works under weaker conditions and extends the result to more general planar domains containing an open wedge.

Much less is known about the situation when the particle is confined from both sides. It is true, of course, that many numerical studies of such systems which model various quantum wires can be found in the physical literature, but rigorous results are scarce. This is our motivation in considering such a potentially confined channel. For the sake of simplicity we suppose that the channel is straight and that the potential is parabolic with constant strength along the axis. This is certainly a reasonable model which has the advantage that it allows us to solve the unperturbed problem analytically. We prove two types of results.

First, if a bounded potential W periodic in the longitudinal direction is added, the bottom of the spectrum remains absolutely continuous for weak enough perturbations. On the one hand, we discuss the number of gaps in such a continuous spectrum as a function of the strength of the confining potential. On the other hand, we prove that if W depends only on the longitudinal or on the transverse variable, the whole spectrum remains absolutely continuous, independently of the strength of the potential.

Second, if the perturbation W is no longer assumed to be periodic, we prove that a part of the spectrum remains absolutely continuous provided W is small in a suitable sense and satisfies a weak 'localization' condition.

Let us describe in more detail the results and contents of this paper. The unperturbed Hamiltonian will be

$$H_0 = H_L(B) + \omega^2 y^2 \tag{1.1}$$

where $H_L(B) = p_y^2 + (p_x + By)^2$ is the free magnetic Hamiltonian with a homogeneous magnetic field B. The last operator corresponds to the Landau gauge, which we will use throughout the paper.

In the following two sections we analyse periodic perturbations, i.e. the structure of the spectrum of

$$H = H_0 + W \tag{1.2}$$

where the potential $W(x, y)$ is ℓ-periodic in x. The periodicity enables us to use the Bloch decomposition and to write the generalized eigenfunctions of H_0 in the form

$$\psi_m(x)\varphi_n(y, m + \theta) \tag{1.3}$$

where $m \in \mathbb{Z}$, $n \in \boldsymbol{N}_0$, and θ is the corresponding Bloch parameter running through the Brillouin zone $[-\pi/\ell, \pi/\ell)$. In the absence of perturbation it is straightforward to see that the spectrum is purely absolutely continuous and includes all energies in the interval $\left[\sqrt{B^2+\omega^2}, \infty\right)$. Perturbation theory then shows that for any $E > 0$, the part of the spectrum inside the interval $\left[\sqrt{B^2+\omega^2} - \|W\|, E\right]$ is still purely absolutely continuous, provided $\|W\|$ is small enough.

Next, using an appropriately modified Thomas argument (cf [Tm] and the generalization in [RS, section XIII.16]) we will prove in theorem 2.1 that the whole spectrum of H remains purely absolutely continuous if $W(x, y) \equiv W(x)$ depends on x only and is essentially bounded. The same is true if $W(x, y) \equiv W(y)$ depends on the transverse variable only and is essentially bounded.

Finally, we address the question about the number of open gaps in the spectrum. One can find a partial answer using properties of the function $W_0 := (\varphi_0, W(\cdot, y)\varphi_0)$ which represents the projection of the potential onto the lowest transverse mode. If the latter is non-constant, the one-dimensional Schrödinger operator $K = -\partial_x^2 + W_0(x)$ on $L^2(\mathbb{R})$ has by [RS, theorem XIII.90] a purely absolutely continuous spectrum with open gaps—at least one but generically infinitely many. We will show in section 3 that these gaps persist in the spectrum of the operator (1.2) provided that the coupling constant of the confinement is large enough, see theorem 3.1. Therefore, such a channel can have generically any finite number of open gaps for any bounded x-periodic perturbation, provided it is confining enough.

Non-periodic perturbations require a different technique. In the last part of this paper, section 4, we address this question in a similar way to that of the papers mentioned above, namely by using a Mourre operator related to a distinguished classical quantity. Recall that the central point of the Mourre theory is to find a suitable self-adjoint conjugate operator A such that in certain states the expectation value of $[H_0 + W, \mathrm{i}A]$ will have a definite sign. Classically, it amounts to finding an observable increasing in time. This motivated the choice of the conjugate operator in [BP, FGW] where the classical particle followed the boundary counterclockwise and therefore propagated in a definite direction. Accordingly, the coordinate parallel to the boundary gave a conjugate operator with the needed properties.

By contrast, in our case there are two 'boundaries' which allow for classical motion in both directions along the x axis. Of course, they are edges with a grain of salt, since their 'distance' depends on the particle energy.

Little is presently known about the stability of transport in systems without a preferred direction. The existing results always assume in some form that the 'opposite' edge currents can be placed at arbitrarily large distance to prevent their destructive interference. This is the case for domains containing wedges in [FGW] which we mentioned earlier. Another example is the recent paper [FM], which studies the nature of the spectrum of the random Schrödinger operator with magnetic field in a finite macroscopic system. The particle is supposed to be confined in one direction by two smooth boundaries separated by a distance equal to L, and the other direction is L periodic. It is then shown that for L large enough there exist realizations of random potentials such that the spectrum in the vicinity of Landau levels contains both current carrying states and localized states. Roughly speaking, this is due to decoupling of bulk and edge states in the limit of large L. It is also announced, that away from the Landau levels there are current carrying states only. Notice that the transverse distance in [FM] may grow slower, say as L^α with $\alpha \in (0, 1)$, but it cannot be kept constant.

In models of a channel with a fixed cross section there is no external parameter to control the decoupling, and it is not *a priori* clear how the spectrum will behave. We start the Mourre analysis by solving the classical problem in the absence of the potential W. The trajectories turn out to be drifting ellipses. We take the x-coordinate of the ellipse centre multiplied by the corresponding momentum component as the quantity to determine the conjugate operator. This allows us to find that under suitable smallness assumptions about W there are intervals separated from the modified Landau levels where the spectrum contains no eigenvalues or even, under stronger hypothesis on W, remains absolutely continuous. Due to the destructive interference between the opposite 'edge' currents our conditions on the disorder potential include, in addition to the finiteness of $\sup|W(x, y)|$ required in [BP, FGW], also a sort of localization requirement. In particular, we need $\sup|x\partial_x W(x, y)|$ respectively $\sup|x^2\partial_x^2 W(x, y)|$ to be finite. Of course, many 'non-local' potentials fit in, say those with different limits as $x \to \pm\infty$, and any power-like decay at large $|x|$ will do, however, the said condition excludes the most typical random potentials in the form of a sum of randomly placed copies of a single-impurity potential. For such potentials we establish the existence of transport only in the situation when the 'dirty' part of the channel has a finite length, see theorem 4.3. We also discuss the behaviour of our model in the limit of strong confinement, i.e. when $\omega \to \infty$.

More than that, we show in section 4.3 that any Mourre operator quadratic in the canonical variables will lead here to the same restriction. Hence an attempt to establish for a 'fixed-width' channel a result comparable to [BP, FGW] by the conjugate-operator method has to employ another A. Obvious candidates are those which combine the first-order canonical variable with a (sign-changing) localization of the particle in the vicinity of the edges. However, attempts in this direction which we are aware of have not been successful so far and the problem remains open.

2. Periodic perturbations

In this section we first give explicit expressions for the eigenvalues and eigenfunctions of H_0, which is possible due to the specific choice of our confinement potential. Then, as mentioned above, we will investigate the nature of the spectrum when we add a periodic perturbation.

The Hamiltonian of the system we are interested in is thus of the following form:

$$H = -\partial_y^2 + (-\mathrm{i}\partial_x + yB)^2 + \omega^2 y^2 + W(x, y) \qquad \text{on} \quad L^2(\mathbb{R}^2) \tag{2.1}$$

where W is bounded and ℓ-periodic in x. The scaling

$$x, y \to \lambda x, \lambda y \qquad B \to \lambda^{-2} B \qquad \omega \to \lambda^{-2}\omega \qquad W \to \lambda^{-2} W$$

gives $H \to \lambda^{-2}H$. Without loss we can thus assume $\ell = 2\pi$. By [RS, theorem 10.34], H is e.s.a. on $C_0^\infty(\mathbb{R}^2)$. We use the periodicity of W and apply the Bloch decomposition in x writing

$$H = \int_{|\theta|\leqslant 1/2}^{\oplus} H(\theta)\,\mathrm{d}\theta \tag{2.2}$$

where $H(\theta)$ has the form (2.1) on $L^2([0, 2\pi] \times \mathbb{R})$ with the boundary conditions

$$\partial_x^j \psi(2\pi-, y) = \mathrm{e}^{\mathrm{i}\theta 2\pi}\partial_x^j \psi(0+, y) \qquad j = 0, 1. \tag{2.3}$$

Let us now turn to the properties of the fibre operator

$$\tilde{H}_0(\theta) = -\partial_y^2 + (-\mathrm{i}\partial_x + yB)^2 + \omega^2 y^2. \tag{2.4}$$

After transferring θ from the boundary conditions to the operator we find that $\tilde{H}_0(\theta)$ is unitarily equivalent to

$$H_0(\theta) = (-\mathrm{i}\partial_x + By + \theta)^2 - \partial_y^2 + \omega^2 y^2 \qquad \text{on} \quad L^2([0, 2\pi] \times \mathbb{R}) \tag{2.5}$$

with periodic boundary conditions at $x = 0$ and 2π. We exhibit below a complete set of eigenvectors in

$$D_e \equiv \{f \in W^{2,2}([0, 2\pi]) | f(0) = f(2\pi), f'(0) = f'(2\pi)\} \otimes S(\mathbb{R}) \tag{2.6}$$

where $S(\mathbb{R})$ denotes the set of the Schwarz function, showing that $H_0(\theta)$ is essentially self-adjoint on D_e. Next we show that $H_0(\theta)$ is a holomorphic family of type A in the sense of Kato. Let $H_0(0)$ be self-adjoint on its domain D and let us formally expand the operator $H_0(\theta)$ as

$$H_0(\theta) = (-\mathrm{i}\partial_x + By)^2 - \partial_y^2 + \omega\, y^2 + 2\theta(-\mathrm{i}\partial_x + By) + \theta^2. \tag{2.7}$$

We note that $(-\mathrm{i}\partial_x + By)$ is symmetric on D_e and denote the resolvent by $R_0(\theta, z) = (H_0(\theta) - z)^{-1}$. Now, for any $\varphi \in D_e$

$$\begin{aligned} \|(-\mathrm{i}\partial_x + By)\varphi\|^2 &\leqslant \langle\varphi|H_0(0)\varphi\rangle = \langle R_0(0,z)(H_0(0) - z)\varphi|H_0(0)\varphi\rangle \\ &\leqslant \|R_0(0,z)\|\|H_0(0)\varphi\|^2 + |z|\langle\varphi|R_0(0,\bar{z})H_0(0)\varphi\rangle \\ &\leqslant C(z)\|H_0(0)\varphi\|^2 + |z|^2C(z)\|\varphi\|^2 \end{aligned} \tag{2.8}$$

where $C(z) = \mathrm{O}(1/\mathrm{Im}\, z)$, as $\mathrm{Im}\, z \to \infty$, $|\mathrm{Re}\, z| < \infty$. From theorem 5.4.4 p 288 in [Ka], we deduce that that $(-\mathrm{i}\partial_x + By)$ is relatively bounded with respect to $H_0(0)$ on D_e, with arbitrarily small relative bound (to this end, take $|\mathrm{Im}\, z|$ large enough). Hence the domain of $H_0(\theta)$ coincides with D for any complex θ, and the expansion (2.7) shows that the vector $H_0(\theta)\psi$ is holomorphic in θ for any $\psi \in D$. That means $H_0(\theta)$ is a self-adjoint holomorphic family of type A in the whole complex plane, see [Ka], pp 375 and 385. The same is true for the perturbed operator

$$H(\theta) = H_0(\theta) + W(x, y) \tag{2.9}$$

when W is bounded.

In order to find the spectrum of $H_0(\theta)$ we introduce the basis

$$\psi_m(x) = (2\pi)^{-1/2} \exp(\mathrm{i}mx) \tag{2.10}$$

and get the decomposition

$$\begin{aligned} H_0(\theta) &= \bigoplus_{m\in \boldsymbol{Z}} |\psi_m\rangle H_0^m(\theta)\langle\psi_m| \\ &= \bigoplus_{m\in \boldsymbol{Z}} |\psi_m\rangle\big[(m + By + \theta)^2 - \partial_y^2 + \omega^2y^2\big]\langle\psi_m| \end{aligned} \tag{2.11}$$

where $H_0^m(\theta) = \langle\psi_m|H_0(\theta)\psi_m\rangle$. By a unitary transform inducing a $(\theta + m)$-dependent shift of the argument we find that $H_0^m(\theta)$ is unitarily equivalent to

$$\tilde{H}_m(\theta) = -\partial_u^2 + \alpha^2u^2 + \beta(m + \theta)^2 \tag{2.12}$$

with $\alpha = \sqrt{B^2 + \omega^2}$, $\beta = \omega^2/(B^2 + \omega^2)$, and $u = y + B(m + \theta)/(B^2 + \omega^2)$. This operator is clearly analytic in θ. Therefore we get the spectrum

$$\sigma(H_0^m(\theta)) = \big\{\alpha(2n + 1) + \beta(m + \theta)^2\big\} = \{E_n(\theta + m)\}_{n\in \boldsymbol{N}_0} \tag{2.13}$$

where the corresponding eigenfunctions of $H_0^m(\theta)$, $\varphi_n^{m+\theta}(y)$, are translates of the usual harmonic oscillator states $\varphi_n(u)$. More precisely, if $V_{\theta+m}$ is the unitary operator from $L^2(\mathbb{R}_y)$ to $L^2(\mathbb{R}_u)$ defined by

$$(V_{\theta+m}f)(u) = f(u - B(m + \theta)/(B^2 + \omega^2)) \tag{2.14}$$

then $V_{\theta+m}H_0^mV_{\theta+m}^{-1} = \tilde{H}_m(\theta)$ and $\varphi_n^{m+\theta}(y) = (V_{\theta+m}^{-1}\varphi_n)(y)$. For a later purpose, let us also introduce the unitary operator $V(\theta)$ from $L^2(\mathbb{R}_x \times \mathbb{R}_u)$ to $L^2(\mathbb{R}_x \times \mathbb{R}_y)$ given as

$$V(\theta) = \bigoplus_{m\in \boldsymbol{Z}} V_{\theta+m}. \tag{2.15}$$

Let us turn to

$$H(\theta) = H_0(\theta) + W(x, y) \qquad \text{on} \quad L^2([0, 2\pi] \times \mathbb{R}) \tag{2.16}$$

with periodic boundary conditions at $x = 0$ and 2π. Since W is bounded, it is relatively compact w.r.t. $H_0(\theta)$ and the essential spectrum of $H(\theta)$ is thus the same as that of $H_0(\theta)$. It follows that $\sigma(H(\theta))$ is discrete. The corresponding eigenvalues are analytic functions of θ, we denote them as $E_j(\theta)$.

At this point, we see that for any $E' > 0$, and uniformly in $|\theta| < 1/2$, there are finitely many eigenvalues of $H_0(\theta)$ $E_{n,m}(\theta) = \alpha(2n+1) + \beta(m+\theta)^2$ below E'. These eigenvalues are branches of analytic functions in θ and may display finitely many crossings with one another. The same is true for those of the perturbed operator $H(\theta)$. In order to exclude the possibility for a perturbed eigenvalue to be constant in θ, it is enough to impose that the perturbation be smaller than half the smallest variation of the finitely many arcs of analytic functions free from crossings below E. Therefore, below $E = E' - \|W\|$, the eigenvalues of $H(\theta)$ cannot be constant and we have

Proposition 2.1. *For any $E > 0$, the spectrum of the Hamiltonian* (2.1) *is purely absolutely continuous below E if $\|W\|_\infty$ is small enough.*

Let us turn to the case where W depends on x only. We are interested in the properties of the eigenvalues of $H_0^m(\theta)$, which coincide with those of $\tilde{H}_m(\theta)$. As the eigenfunctions of $\tilde{H}_m(\theta)$ are independent of $m + \theta$, it is easier to deal with this operator as θ becomes complex than with $H_0^m(\theta)$. We define

$$h_0(\theta) := V(\theta)\, H_0(\theta)\, V^{-1}(\theta) \tag{2.17}$$

then we have the relation

$$\|(h_0(\theta)+1)^{-1}\|^2 = \sup_{m\in\mathbf{Z}} \|r_m(\theta) r_m(\theta)^*\| \qquad r_m(\theta) := (\tilde{H}_m(\theta)+1)^{-1}. \tag{2.18}$$

When θ becomes complex, in which case we will write $\theta = \theta_1 + \mathrm{i}\,\theta_2$, the resolvent $r_m(\theta)$ remains compact and $r_m(\theta)^* = (\tilde{H}_m(\overline{\theta})+1)^{-1}$ so that

$$\|r_m(\theta) r_m(\theta)^*\| = \sup_{n\in\mathbf{N}_0} \frac{1}{|E_n(\theta+m)+1|^2} \tag{2.19}$$

since the basis $\{\varphi_n(u)\}_{n\in\mathbf{N}_0}$ remains orthonormal for complex θ. Then one can show that this norm goes to zero as $\theta \to \infty$ in some direction of the complex plane, uniformly in $m \in \mathbb{Z}$. Indeed, from (2.13) we get

$$\begin{aligned}\|r_m(\theta) r_m(\theta)^*\| &= \sup_{n\in\mathbf{N}_0} \frac{1}{[\alpha(2n+1)+\beta((m+\theta_1)^2-\theta_2^2)+1]^2 + [2\beta\theta_2(m+\theta_1)]^2} \\ &\leqslant \frac{1}{[2\beta\theta_2(m+\theta_1)]^2}\end{aligned} \tag{2.20}$$

which goes to zero as $\theta_2 \to \infty$ uniformly in m provided θ_1 is not an integer.

Furthermore, from the fact that $h_0(\theta)$ is a self-adjoint holomorphic family of type A it follows that $(h_0(\theta)+1)^{-1}$ is compact either for all θ or for no θ, cf [Ka, theorem VII.2.4]. We have already seen that $(h_0(\theta)+1)^{-1}$ is compact for θ real, so it is compact also for θ complex. Thus $(h_0(\theta)+1)^{-1}\,(h_0^*(\theta)+1)^{-1}$ is a compact self-adjoint operator, and since the family $\{\varphi_n(u)\}_{n\in\mathbf{N}_0}$ still forms a complete orthonormal basis in $L^2(\mathbb{R})$, the eigenvalues of $h_0(\theta)$ retain the form (2.13). Hence one has

$$\|(h_0(\theta)+1)^{-1}(h_0(\overline{\theta})^*+1)^{-1}\| = \|(h_0(\theta)+1)^{-1}\|^2 \leqslant \frac{1}{\beta^2\theta_2^2} \tag{2.21}$$

where we have chosen for simplicity $\theta_1 = 1/2$.

The perturbed fibre operator is

$$h(\theta) = h_0(\theta) + V(\theta)\, W(x)\, V^{-1}(\theta) = h_0(\theta) + W(x). \tag{2.22}$$

The point is now to show that the eigenvalues $E_j(\theta)$ of $h(\theta)$ are not constant in θ. Then the same is true, as for θ real, also for the eigenvalues of

$$H(\theta) = H_0(\theta) + W(x, y) \tag{2.23}$$

and this yields the absolute continuity of (2.1).

We use the Thomas argument ([Tm] and [RS, section XIII.16]) and assume that some $E_j(\theta)$ is equal to E_0 for all θ. From the above analysis it follows that E_0 is an eigenvalue of $h(\theta)$ also for all complex θ, and therefore

$$\|(h(\theta) + 1)^{-1}\| \geqslant (E_0 + 1)^{-1}. \tag{2.24}$$

On the other hand, a standard argument based on the resolvent identity shows that for $\|W(x)(h_0(\theta) + 1)^{-1}\| < 1$ (i.e. θ_2 large enough, cf (2.21)) is

$$\|(h(\theta) + 1)^{-1}\| \leqslant \frac{\|(h_0(\theta) + 1)^{-1}\|}{1 - \|W(x)(h_0(\theta) + 1)^{-1}\|} \tag{2.25}$$

so $\|(h(\theta) + 1)^{-1}\| \to 0$ as $\theta_2 \to \infty$ by (2.21). In this way we get a contradiction with (2.24), so no $E_j(\cdot)$ can be constant.

Finally, we note also that if $W(x, y) \equiv W(y)$ is bounded and depends on y only, we get by simple manipulations that H is unitarily equivalent to

$$H \simeq \int_{p\in R}^{\oplus} H(p)\,\mathrm{d}p \tag{2.26}$$

where

$$H(p) = -\partial_y^2 + \alpha^2 y^2 + p^2 \frac{\omega^2}{B^2 + \omega^2} + W(y - p\,B/(B^2 + \omega^2)). \tag{2.27}$$

As W is bounded, we see that the analytic eigenvalues $\{e_n(p)\}_{n\in N}$ of $H(p)$ tend to $\alpha(2n + 1) + p^2 \frac{\omega^2}{B^2+\omega^2}$ as $p \to \infty$. Therefore they cannot be constant and the spectrum of H is purely absolutely continuous also.

This allows us to make the following claim.

Theorem 2.1. *Let $W_1(x) \in L^\infty(\mathbb{R})$ be periodic in x and $W_2(y) \in L^\infty(\mathbb{R})$. Then the spectra of both operators*

$$H = -\partial_y^2 + (-\mathrm{i}\partial_x + yB)^2 + \omega^2 y^2 + W_1(x) \tag{2.28}$$

$$H = -\partial_y^2 + (-\mathrm{i}\partial_x + yB)^2 + \omega^2 y^2 + W_2(y) \tag{2.29}$$

are purely absolutely continuous for any $\omega \neq 0$.

3. Open gaps

The result of the previous section shows that the absolute continuity of the bottom of the spectrum of the magnetic Hamiltonian in the presence of a parabolic confinement is not affected by a small bounded x-periodic perturbation. Of course, one would like to know how the spectrum looks like as a set, in particular how many gaps can open as a consequence of the perturbation. We now show that for a non-constant $W(\cdot, y)$ there are generically many gaps in the spectrum of H provided the coupling constant of the confinement is large enough.

We start again with the fibre Hamiltonian

$$H(\theta) = -\partial_y^2 + (-\mathrm{i}\partial_x + yB)^2 + \omega^2 y^2 + W(x, y) \tag{3.1}$$

on $L^2([0, 2\pi] \times \mathbb{R})$ with the boundary conditions (2.3). We introduce a new variable s by

$$s = \sqrt{\alpha}\, y \qquad \alpha := \sqrt{B^2 + \omega^2} \tag{3.2}$$

and the orthonormal basis on $L^2(\mathbb{R})$

$$\varphi_n(s) = C_n \exp(-s^2/2)\, H_n(s) \qquad C_n = (1/\pi)^{1/4}\,(2^n n!)^{-1/2} \qquad n \in \boldsymbol{N_0}. \tag{3.3}$$

Let us introduce some more notations:

$$\begin{aligned} W_{n,m}^{(\alpha)}(x) &= (\varphi_n, W\varphi_m) = \int_{\mathbb{R}} \varphi_n(s)\,\varphi_m(s) W(x, s/\sqrt{\alpha})\,\mathrm{d}s \qquad n \neq m \\ W_n^{(\alpha)}(x) &= (\varphi_n, W\varphi_n) = \int_{\mathbb{R}} \varphi_n(s)\varphi_n(s) W(x, s/\sqrt{\alpha})\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

The matrix elements of $H(\theta)$ in the basis (3.3) are then the operators on $L^2([0, 2\pi])$ given by

$$\begin{aligned} H_{n,m}(\theta) &= \delta_{n,m}\,[\alpha(2n+1) + K_n(\theta)] + W_{n,m}^{(\alpha)}(x)(1 - \delta_{n,m}) \\ &\quad -\delta_{n+1,m}\sqrt{\frac{2(n+1)}{\alpha}}\,\mathrm{i}\,B\partial_x - \delta_{n-1,m}\sqrt{\frac{2n}{\alpha}}\,\mathrm{i}B\partial_x \end{aligned} \tag{3.5}$$

where we define $K_n(\theta)$ as

$$K_n(\theta) = -\partial_x^2 + W_n^{(\alpha)}(x) \tag{3.6}$$

with the domain

$$D(\theta) = \left\{ f \in W_{2,2}[0, 2\pi];\ f(2\pi) = \mathrm{e}^{2\pi\mathrm{i}\theta} f(0),\ f'(2\pi) = \mathrm{e}^{2\pi\mathrm{i}\theta} f'(0) \right\}.$$

By [RS, section XIII.16] for each $n \in \boldsymbol{N_0}$ the operator $K_n(\theta)$ has a purely discrete spectrum, and none of their eigenvalues is constant in θ. We will denote the eigenvalues and eigenfunctions of $K_n(\theta)$ by

$$\epsilon_k(n, \theta);\ \psi_k^n(x, \theta) \qquad k \in \mathbb{Z} \tag{3.7}$$

respectively, where for any fixed θ and n the functions $\psi_k^n(x, \theta)$ form an orthonormal basis in $L^2[0, 2\pi]$. It is shown in [RS, theorem XIII.91] that for a non-constant W_n at least one gap is present in the spectrum of

$$K_n := \int_{|\theta| \leqslant 1/2}^{\oplus} K_n(\theta)\,\mathrm{d}\theta.$$

In other words, there exists some j such that

$$\sup_{|\theta| \leqslant 1/2} \epsilon_j(n, \theta) < \inf_{|\theta| \leqslant 1/2} \epsilon_{j+1}(n, \theta). \tag{3.8}$$

We are particularly interested in the spectrum of $H_{0,0}$, the direct integral from $H_{0,0}(\theta)$ over θ, which contains at least one gap if $W_0^{(\alpha)}$ is not constant.

It follows from (3.5) that taking α large enough, this gap will not be covered by the spectra of the other diagonal elements of $H_{n,m}(\theta)$. Then one needs only show that this gap remains open after taking into account the off-diagonal elements of $H_{n,m}(\theta)$. To see that, we apply perturbation theory. As the unperturbed operator we take

$$H^D(\theta) = \bigoplus_{n \in \boldsymbol{N_0}} H_{n,n}(\theta) \qquad \text{on} \quad L^2[0, 2\pi] \times l_2 \tag{3.9}$$

with eigenvalues and eigenvectors given by

$$\alpha(2n+1) + \epsilon_k(n, \theta) \qquad \psi_k^n(x, \theta) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \tag{3.10}$$

respectively, where 1 stands in the nth row. Moreover, we have the following.

Lemma 3.1. *Let* $H^{OD}(\theta) = H(\theta) - H^D(\theta)$. *Then*

$$\|H^{OD}(\theta)(H^D(\theta) + \mathrm{i})^{-1}\| = \mathcal{O}(1/\alpha) \qquad \text{as} \quad \alpha \to \infty \tag{3.11}$$

uniformly in θ.

Proof. For

$$W^D = \bigoplus_{n \in \mathbf{N_0}} W_n^\alpha(x)$$

we define $W^{OD} = W - W^D$. Then

$$\|W^{OD}(H^D(\theta) + \mathrm{i})^{-1}\| \leqslant 2\|W\|_\infty \|(H^D(\theta) + \mathrm{i})^{-1}\| = \mathcal{O}(1/\alpha) \tag{3.12}$$

as $\alpha \to \infty$ since $\mathrm{dist}(\sigma(H^D(\theta)), \mathrm{i})$ grows linearly with α.

Let us now take n fixed. For the other elements of $H^{OD}(\theta)$, i.e. the last two terms on the rhs of (3.5), we have

$$\left(\mathrm{i}\partial_x \pm \sqrt{\alpha(2n+1)}\right)^2 > 0 \qquad \pm 2\mathrm{i}\sqrt{\alpha(2n+1)}\,\partial_x \leqslant -\partial_x^2 + \alpha(2n+1) \tag{3.13}$$

so that as quadratic forms on $D(\theta)$

$$-\frac{B^2}{\alpha}\,2(n+1)\,\partial_x^2 \leqslant \frac{B^2}{\alpha^2}\,(-\partial_x^2 + \alpha(2n+1))^2. \tag{3.14}$$

Then, in the sense of (3.11),

$$\begin{aligned}
&\||\psi_n\rangle\langle\psi_n|\mathrm{i}B\alpha^{-1/2}\sqrt{2(n+1)}\,\partial_x\,|\psi_{n+1}\rangle\langle\psi_{n+1}|(H^D(\theta) + \mathrm{i})^{-1}\| \\
&\quad = \|\mathrm{i}B\alpha^{-1/2}\sqrt{2(n+1)}\,\partial_x\,(H_{n+1,n+1}(\theta) + \mathrm{i})^{-1}\| \\
&\quad \leqslant \frac{B}{\alpha}\|(-\partial_x^2 + \alpha(2n+1))(-\partial_x^2 + W_{n+1}^\alpha \alpha(2n+3) + \mathrm{i})^{-1}\| \\
&\quad \leqslant \frac{B}{\alpha}\left(1 + \|W_{n+1}^\alpha(-\partial_x^2 + W_{n+1}^\alpha + \alpha(2n+3) + \mathrm{i})^{-1}\|\right) = \mathcal{O}(1/\alpha)
\end{aligned} \tag{3.15}$$

as $\alpha \to \infty$, uniformly in n. Inequality (3.12) and the Schur condition, [Ka, example 3.2.3], then give the statement of the lemma. □

Now the resolvent identity in combination with (3.11) implies

$$\begin{aligned}
&\|(H(\theta) + \mathrm{i})^{-1} - (H^D(\theta) + \mathrm{i})^{-1}\| \\
&\quad = \|(H(\theta) + \mathrm{i})^{-1} H^{OD}(\theta)(H^D(\theta) + \mathrm{i})^{-1}\| \to 0 \qquad \text{as} \quad \alpha \to \infty
\end{aligned} \tag{3.16}$$

so that $H^D(\theta)$ converges to $H(\theta)$ in norm resolvent sense, uniformly in θ. From perturbation theory, see [Ka, theorem IV.2.25], we thus get the convergence of the spectra of $H^D(\theta)$ and $H(\theta)$. It follows that for large enough α, keeping B fixed, the gap between $\epsilon_j(0, \theta)$ and $\epsilon_{j+1}(0, \theta)$ will be open also in the spectrum of H. The argument works for any fixed $j \in \mathbb{Z}$, i.e. sending $\alpha \to \infty$ we can keep any finite family of gaps contained in $\sigma(H_{0,0})$ open. We have thus proven:

Theorem 3.1. *Let* $W(x, y) \in L^\infty(\mathbb{R}^2)$. *Denote by* $N(H)$ *and* $N(H_{0,0})$ *the number of open gaps in the spectrum of* H *and* $H_{0,0}$, *respectively. If* $N(H_{0,0})$ *is finite, then* $N = N(H_{0,0})$ *holds for* ω *large enough; in particular, an open gap exists for a sufficiently strong confinement whenever the function* W_0 *is non-constant. If* $N(H_{0,0}) = \infty$, *then to any positive integer* n *there is* $\omega(n)$ *such that*

$$N(H) \geqslant n$$

holds for all $\omega \geqslant \omega(n)$.

Remark. It is also clear from the above given argument, that taking ω large enough gives us the absolute continuity of $\sigma(H)$ at the bottom of the spectrum. More precisely, in the interval $[\inf \sigma(H_{0,0}),\ \inf \sigma(H_{1,1})]$.

4. Transport in the presence of localized perturbations

As we have indicated in the introduction, we turn now to situations when the perturbation is not periodic, but bounded and localized in a sense to be precised below. In this case we have

$$H = H_0 + W = -\partial_y^2 + (-\mathrm{i}\partial_x + yB)^2 + \omega^2 y^2 + W(x, y) \qquad \text{on} \quad L^2(\mathbb{R}^2) \tag{4.1}$$

with $W(x, y) \in L^\infty(\mathbb{R}^2)$. By [RS, ch X] the Hamiltonian (4.1) is e.s.a. on $C_0^\infty(\mathbb{R}^2)$. For later purposes we notice that $S(\mathbb{R}^2)$, the Schwarz functions, is also a core for H. This follows from the fact that H is clearly symmetric on $S(\mathbb{R}^2)$ and $C_0^\infty(\mathbb{R}^2)$ is included in $S(\mathbb{R}^2)$. The question is the following: in what part of the spectrum and under which conditions does transport survive in the presence of the impurity potential $W(x, y)$?

Instead of the Bloch decomposition we now employ the commutator method. The point is to find a suitable conjugate operator A which satisfies the Mourre estimate

$$E_\Delta(H)[H, \mathrm{i}A]E_\Delta(H) \geqslant \kappa E_\Delta(H) \tag{4.2}$$

for some strictly positive constant κ. Here $E_\Delta(H)$ is the spectral projection of H on the interval Δ. Then, under some regularity assumptions on H, we can obtain the absence of the point spectrum in the interval Δ using the Virial theorem, [GG].

Theorem 4.1 (Virial). *Let H, A be self-adjoint operators on $L^2(\mathbb{R}^2)$ and assume that H is of class $C^1(A)$, i.e. there is $z \in \rho(H)$ such that*

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}tA}(z - H)^{-1}\mathrm{e}^{-\mathrm{i}tA} \tag{4.3}$$

is of class C^1 in the strong operator topology. Then

$$(\psi, [H, \mathrm{i}A]\psi) = 0$$

for any eigenfunction ψ of H.

Under stronger hypothesis on H, we can apply the Mourre theorem (cf [Mo, ABG]) and exclude even the possibility of singular continuous spectrum in Δ. For a precise statement of the Mourre theorem, we have the formulation from [Sa1, Sa2].

Theorem 4.2 (Mourre). *Let H, A be self-adjoint operators on $L^2(\mathbb{R}^2)$ and assume that:*

(1) There is $\alpha > 0$ such that H is of class $C^{1+\alpha}(A)$, i.e. H is $C^1(A)$ and the derivative of (4.3) is Hölder continuous of order α.
(2) H and A satisfy the estimate (4.2) for an open interval Δ and $\kappa > 0$.

Then the spectrum of H in the interval Δ is purely absolutely continuous.

Remark. We shall use the last theorem with $\alpha = 1$, which corresponds to the original formulation given in [Mo], see also [CFKS, theorem 4.9].

The classical counterpart of the positive commutator (4.2) is an observable which increases in time. To find a suitable candidate for the conjugate operator in our case, let us therefore discuss first the classical dynamics of the unperturbed system.

4.1. Classical solution in the absence of perturbation

We will denote the position vector of the particle by $(x(t), y(t))$. In the absence of $W(x, y)$ the classical Hamiltonian is of the form

$$H_{cl} = (p_x + yB)^2 + p_y^2 + \omega^2 y^2 \tag{4.4}$$

where

$$p_x(t) = \tfrac{1}{2}\,\dot{x}(t) - y(t)\,B \qquad p_y(t) = \tfrac{1}{2}\,\dot{y}(t). \tag{4.5}$$

From Hamilton's equations we thus get

$$\dot{p}_x(t) = 0 \qquad \dot{p}_y(t) = -\dot{x}(t)B - 2\omega^2\, y(t). \tag{4.6}$$

Given initial conditions $x(0)$, $y(0)$, $p_x(0)$, $p_y(0)$, the solution of (4.6) reads

$$\begin{aligned}
x(t) &= -\frac{B}{\alpha^2}\, p_y(0)\cos(2\alpha t) + \frac{B}{\alpha}\left(y(0) + \frac{B}{4\alpha^2}\, p_x(0)\right)\sin(2\alpha t) \\
&\qquad + 2p_x(0)\, t\,\frac{\omega^2}{\alpha^2} + x(0) + \frac{B}{\alpha^2}\, p_y(0) \\
y(t) &= \alpha^{-1} p_y(0)\sin(2\alpha t) + \left(y(0) + \frac{B}{4\alpha^2}\, p_x(0)\right)\cos(2\alpha t) - \frac{B}{\alpha^2} p_x(0) \\
p_x(t) &= p_x(0) \\
p_y(t) &= p_y(0)\cos(2\alpha t) - \alpha\left(y(0) + \frac{B}{4\alpha^2}\, p_x(0)\right)\sin(2\alpha t).
\end{aligned} \tag{4.7}$$

Note that the momentum p_x is preserved since the free Hamiltonian H_0 commutes with x-translations, see (4.6). It is easy to see that the classical trajectory is now given by an ellipse, with the position vector of its centre being

$$S(t) = \left[2p_x(0)\, t\,\frac{\omega^2}{\alpha^2} + x(0) + \frac{B}{\alpha^2}\, p_y(0), -\frac{B}{\alpha^2} p_x(0)\right] \tag{4.8}$$

so that as long as $\omega \neq 0$, i.e. the confinement is present, the centre of the ellipse is moving along the x axis with constant velocity and in the direction given by a sign of the initial momentum $p_x(0)$. Note also, that the two ellipses which correspond to the motions in opposite directions are mutually shifted by $\frac{2B}{\alpha^2} p_x(0)$.

A classical observable whose absolute value is increasing in time is thus the x-component of $S(t)$, which can be written as

$$S_x(t) = x(t) + \frac{B}{\alpha^2}\, p_y(t). \tag{4.9}$$

However, since we need something which has a definite sign independently of the initial conditions, we multiply (4.9) by $p_x(t)$; then

$$\partial_t(p_x(t)S_x(t)) = 2p_x^2(0)\frac{\omega^2}{\alpha^2} > 0. \tag{4.10}$$

In other words, the corresponding quantum mechanical conjugate operator can be chosen in the form

$$A = \tfrac{1}{2}(-\mathrm{i}\partial_x x - \mathrm{i}x\partial_x) - \frac{B}{\alpha^2}\,\partial_x\partial_y. \tag{4.11}$$

4.2. Absence of eigenvalues and absolute continuity

Now we are going to show that under some regularity and decay assumptions on W the absolutely continuous spectrum of the free Hamiltonian persists in some parts of the spectrum of H. In particular, this makes scattering on the impurity in our parabolic channel possible.

The conditions we impose on $W(x, y)$ then are as follows:

(a) $W_0 := \|W\|_\infty < \alpha,\ W_0' := \|x\partial_x W\|_\infty < \infty$
(b) $W \in C^2(\mathbb{R}^2)$ and

$$\|\partial_x^2 W\|_\infty < \infty,\ \|\partial_y^2 W\|_\infty < \infty,\ \|\partial_x\partial_y W\|_\infty < \infty,\ \|x^2\partial_x^2 W\|_\infty < \infty.$$

Before looking for the Mourre estimate, we check the regularity of the map (4.3).

First we state an auxiliary lemma, which is proven in the appendix.

Lemma 4.1. *There exists a number c such that:*

(i) $\|\partial_y^2 R_0(\lambda)\| \leqslant c$
(ii) $\|\partial_x^2 R_0(\lambda)\|,\ 2\,\|y\partial_x R_0(\lambda)\|,\ \|y^2 R_0(\lambda)\| \leqslant c\,\frac{1+\alpha^2}{\omega^2}$
(iii) $\|\partial_x\partial_y R_0(\lambda)\| \leqslant c\sqrt{\frac{1+\alpha^2}{\omega^2}}$

where $R_0(\lambda) = (H_0 + \lambda)^{-1},\ \lambda \geqslant 0.$

Now we show that under the assumption (a) one can apply the Virial theorem to a pair of operators $H,\ A$.

Lemma 4.2. *Let* $W(x, y)$ *satisfy the condition* (a). *Then* H *is of class* $C^1(A)$.

Proof. By [GG] and [ABG, theorem 6.3.4] to show that H is $C^1(A)$, it is enough to prove that:

(1) e^{itA} preserves $D(H)$,
(2) There is a constant c such that

$$|(H\varphi, A\varphi) - (A\varphi, H\varphi)| \leqslant c\,(\|H\varphi\|^2 + \|\varphi\|^2) \qquad \varphi \in D(H) \cap D(A).$$

Since W is bounded, the domain of H coincides with that of H_0 and we can thus check the condition (1) only for $D(H_0)$. Let D be a core for H_0. It follows from [ABG, lemma 7.6.5], that to prove (1) it suffices to show, in addition to (2), that:

(i) for $u \in D$ and $t \in \mathbb{R}$, $e^{itA}u \in D$ and $\sup_{|t|\leqslant 1} \|H_0 e^{itA}u\| < \infty$.
(ii) the derivative $\partial_t e^{-itA} H_0 e^{itA} u|_{t=0} \equiv [H_0, iA]u$ exists weakly for each vector $u \in D$.

To begin with, we notice that A being quadratic in momentum and position, we know by [Hag, theorem 3.4] that the unitary propagator $U(t) = e^{-itA}$ is such that

$$U(t) : S(\mathbb{R}^2) \mapsto S(\mathbb{R}^2).$$

Now, $S(\mathbb{R}^2)$ is a core for H_0, so the first part of (i) is satisfied. To see how $U(t)$ acts on the function from $S(\mathbb{R}^2)$, we apply a partial Fourier transformation in y, and denote the transformed operators by $\hat{H}_0$ and $\hat{A}$. It can be directly checked, that for any $\psi(x, y) \in S(\mathbb{R}^2)$

$$e^{-it\hat{A}}\hat{\psi}(x, k) = e^{-t/2}\hat{\psi}\left(e^{-t}x - k\mu(1 - e^{-t}), k\right) \tag{4.12}$$

where $\hat{\psi}(x, k) = \mathcal{F}_y\, \psi(x, y)$ and $\mu := \frac{B}{\alpha^2}$.

A simple calculation then gives

$$e^{-it\hat{A}}\, \hat{H}_0\, e^{it\hat{A}}\hat{\psi}(x, k) = a(t)\,\partial_x^2\,\hat{\psi}(x, k) + b(t)\partial_x\partial_k\hat{\psi}(x, k) - \alpha^2\partial_k^2\,\hat{\psi}(x, k) + k^2\hat{\psi}(x, k) \tag{4.13}$$

where

$$\begin{aligned} a(t) &= -e^{2t}\big(1 + 2Be^t\mu(1 - e^t) + \alpha^2 e^{2t}\mu^2(1 - e^t)^2\big) \\ b(t) &= -e^t\big(2B + 2\alpha^2 e^t\mu(1 - e^t)\big) \end{aligned} \tag{4.14}$$

are both C^∞, so that the second part of (i) and (ii) hold.

Moreover, it is easily seen from (4.12) that $U(t)$ is strongly differentiable on $S(\mathbb{R}^2)$. It follows then from [RS, theorem VIII.10] that A is essentially self-adjoint on $S(\mathbb{R}^2)$.

This allows us to verify the condition (2) only on functions in $S(\mathbb{R}^2)$. First we notice that H can be written as

$$H = \left(-i\partial_x \frac{B}{\alpha} + y\alpha\right)^2 - \beta\partial_x^2 - \partial_y^2 + W(x, y) \tag{4.15}$$

reminding that

$$\beta = \frac{\omega^2}{\alpha^2}.$$

Then for any $\varphi \in S(\mathbb{R}^2)$

$$\begin{aligned} |(H\varphi, A\varphi) - (A\varphi, H\varphi)| &\leqslant |(\varphi, -2\,\beta\,\partial_x^2\,\varphi)| \\ &\quad + \mu|(W\varphi, \partial_x\partial_y\varphi) - (W\varphi, \partial_x\partial_y\varphi)| + |(\varphi, (\partial_x W)x\varphi)| \\ &\leqslant 2|(\varphi, H_0\varphi)| + 2\mu W_0\|\varphi\|\,\|\partial_x\partial_y\varphi\| + \|\varphi\|^2 W_0'. \end{aligned} \tag{4.16}$$

On the other hand we have

$$\begin{aligned} \|i\partial_x\varphi\|^2 &\leqslant \beta^{-1}\,\|\varphi\|\,\|H_0\varphi\| \leqslant \beta^{-1}\,\|\varphi\|(\|H\varphi\| + W_0\|\varphi\|) \\ \|i\partial_y\varphi\|^2 &\leqslant \|\varphi\|\,\|H_0\varphi\| \leqslant \|\varphi\|(\|H\varphi\| + W_0\|\varphi\|) \end{aligned} \tag{4.17}$$

and since $H \geqslant \alpha - W_0 > 0$ holds by assumption, also

$$\|\varphi\| \leqslant (\alpha - W_0)^{-1}\|H\varphi\|. \tag{4.18}$$

Moreover, it follows from lemma 4.1, that

$$\|\partial_x\partial_y\varphi\| \leqslant \text{const}\,\|H_0\varphi\|. \tag{4.19}$$

Using all the inequalities we can find some large enough constant c, depending on α and W_0, such that

$$|(H\varphi, A\varphi) - (A\varphi, H\varphi)| \leqslant c\,(\|H\varphi\|^2 + \|\varphi\|^2) \tag{4.20}$$

thus proving (2).

Finally, (2) in combination with [ABG, lemma 7.6.5] shows that $e^{it\hat{A}}$ preserves $D(\hat{H}_0)$. That is, for any $\psi(x, y) \in D(H_0)$ we have $e^{it\hat{A}}\hat{\psi}(x, k) \in D(\hat{H}_0)$ and

$$e^{itA}\psi(x, y) = \mathcal{F}_y^{-1}\,e^{it\hat{A}}\hat{\psi}(x, k) \in \mathcal{F}_y^{-1}\,D(\hat{H}_0) = D(H_0) \tag{4.21}$$

which completes the proof of the lemma. □

The hypothesis of the Mourre theorem requires a slightly stronger regularity of H. We will impose some additional assumptions on $W(x, y)$.

Lemma 4.3. *Assume* (a) *and* (b). *Then* H *is* $C^2(A)$.

Proof. We will prove the statement of the lemma separately for H_0 and W.
First we prove that H_0 is $C^\infty(A)$. We work in the Fourier picture, as above. Consider

$$\hat{H}_0(t) = e^{-it\hat{A}}\,\hat{H}_0\,e^{it\hat{A}} \tag{4.22}$$

self-adjoint on $D(\hat{H}_0)$ for any $t \in \mathbb{R}$ and, for $\lambda > \|W\| + 1$,

$$\hat{R}_0(t) = \mathrm{e}^{-\mathrm{i}t\hat{A}}\,(\hat{H}_0 + \lambda)^{-1}\mathrm{e}^{\mathrm{i}t\hat{A}}. \tag{4.23}$$

As $\hat{R}_0(t+t_0) = \mathrm{e}^{-\mathrm{i}t_0\hat{A}}\,\hat{R}_0(t)\mathrm{e}^{\mathrm{i}t_0\hat{A}}$, it is enough to check differentiability at 0. From the resolvent identity on $(\hat{H}_0 + 1)S(\mathbb{R}^2)$ and (4.13), we get

$$\begin{aligned}\hat{R}_0(t) - \hat{R}_0(0) &= -\hat{R}_0(t)(\hat{H}_0(t) - \hat{H}_0)\hat{R}_0(0)\\ &= \hat{R}_0(t)(\tilde{a}(t)\partial_x^2 + \tilde{b}(t)\partial_x\partial_k)\hat{R}_0(0)\\ &\equiv \hat{R}_0(t)B(t)\end{aligned} \tag{4.24}$$

where $\tilde{a}(t)$ and $\tilde{b}(t)$ are both C^∞ and $\mathcal{O}(t)$ as $t \to 0$. It is proven in the appendix, see lemma 4.1, that $\partial_x^2\hat{R}_0(0)$ and $\partial_x\partial_k\hat{R}_0(0)$ are bounded. Therefore the operator $B(t)$ is bounded, C^∞ and $B(t) \to 0$ in norm as $t \to 0$.

With the properties of $B(t)$ listed above, we deduce that in a neighbourhood of $t = 0$

$$\hat{R}_0(t) = \hat{R}_0(0)(\mathbb{I} - B(t))^{-1} \tag{4.25}$$

which is C^∞ in norm, since B is, and we can conclude that H_0 is $C^\infty(A)$.
To show that $(H_0 + W) \in C^2(A)$ it is sufficient by [Mo, CFKS, theorem 4.9] and lemma 4.2 to find some $c > 0$ such that

$$(\varphi, [[W, \mathrm{i}A], \mathrm{i}A]\varphi) \leqslant c(\|H\varphi\|^2 + \|\varphi\|^2) \tag{4.26}$$

for any $\varphi \in D(H) \cap D(A)$. Expanding the second commutator in (4.26) we write for any $\varphi \in S(\mathbb{R}^2)$

$$\begin{aligned}(\varphi, [[W, \mathrm{i}A], \mathrm{i}A]\varphi) &= (\varphi, x(\partial_x W)\varphi) + (\varphi, x^2(\partial_x^2 W)\varphi)\\ &\quad + \mathrm{i}\,\mu\,[2(x(\partial_x W)\varphi, \partial_x\partial_y\varphi) - 2(\partial_x\partial_y\varphi, x(\partial_x W)\varphi)]\\ &\quad + \mathrm{i}\mu\,[(\partial_x\partial_y\varphi, W\varphi) - (\varphi, W\partial_x\partial_y\varphi)] - \mu^2((\partial_x\partial_y W)\varphi, \partial_x\partial_y\varphi)\\ &\quad - \mu^2[(\partial_x\partial_y\varphi, (\partial_x\partial_y W)\varphi) - (\partial_x, (\partial_y^2 W)\partial_x\varphi) - (\partial_y\varphi, (\partial_x^2 W)\partial_y\varphi)].\end{aligned} \tag{4.27}$$

Now we can follow the proof of lemma 4.2 and using the assumption (*b*) we get the following bound:

$$\begin{aligned}|(\varphi, [[W, \mathrm{i}A], \mathrm{i}A]\varphi)| &\leqslant \|\varphi\|^2\|x^2\partial_x^2\,W\|_\infty + W_0'\,\|\varphi\|(\|\varphi\| + 4\|\partial_x\partial_y\varphi\|)\\ &\quad + 2\,\mu\,W_0\|\varphi\|\,\|\partial_x\partial_y\varphi\| + \mu^2\|\partial_y^2 W\|_\infty\|\partial_x\varphi\|^2\\ &\quad + \mu^2\|\partial_x^2 W\|_\infty\|\partial_y\varphi\|^2 + 2\,\mu^2\|\partial_x\partial_y W\|_\infty\|\partial_x\partial_y\varphi\|\,\|\varphi\|\\ &\leqslant \mathrm{const}\,(\|H\varphi\|^2 + \|\varphi\|^2)\end{aligned} \tag{4.28}$$

where the last inequality is justified by lemma 4.1. □

In order to prove the Mourre estimate (4.2) we will proceed in two steps. First, we find a positive lower bound on the contribution to the commutator coming from H_0. Secondly, we control the contribution from W so that we preserve the sought positivity of $[H_0 + W, \mathrm{i}A]$. The former is done in the following.

Lemma 4.4. *Let $\alpha > \delta > 0$ and define*

$$I(\alpha, \delta) := \bigcup_{n\in N_0}[(2n+1)\alpha - \delta, (2n+1)\alpha + \delta]. \tag{4.29}$$

Then for any $E \notin I(\alpha, \delta)$ there exists an open interval $\Delta \ni E$ such that

$$E_\Delta(H)[H_0, \mathrm{i}A]E_\Delta(H) \geqslant \delta E_\Delta(H)$$

holds for W_0 small enough.

Proof. We define an operator

$$H_L(\alpha) = \left(-\mathrm{i}\partial_x \frac{B}{\alpha} + y\alpha\right)^2 - \partial_y^2 \tag{4.30}$$

which is unitarily equivalent to the Landau Hamiltonian with the magnetic field of a strength α, so that $\sigma(H_L(\alpha)) = \{(2n+1)\alpha\}_{n\in N_0}$. It follows that

$$[H_0, \mathrm{i}A] = -2\beta\, \partial_x^2 = 2(H_0 - H_L(\alpha)). \tag{4.31}$$

Now, fix $\lambda \notin I(\alpha, \delta)$ and let us denote by $n_0(\lambda)$ the largest natural number for which $\alpha(2n_0(\lambda)+1) \leqslant \lambda$. The spectral family of H_0 is thus given by

$$E_0(\lambda) = \sum_{n\leqslant n_0(\lambda)} P_n\, \chi_t([0, \lambda - \alpha(2n+1))) \tag{4.32}$$

where P_n is the projection on the nth Landau level of $H_L(\alpha)$ and χ_t is the spectral projection of $-\beta\partial_x^2$.

To continue consider an open interval $\tilde{\Delta} = (E-\epsilon, E+\epsilon)$ with ϵ such that $\tilde{\Delta} \not\subset I(\alpha, \delta)$. For the spectral projection of H_0 on the interval $\tilde{\Delta}$ we then get

$$\begin{aligned} E_{\tilde{\Delta}}(H_0) &= E_0(E+\epsilon) - E_0(E-\epsilon) \\ &= \sum_{n\leqslant n_0(E)} P_n\, \chi_t([E-(2n+1)\alpha-\epsilon, E-(2n+1)\alpha+\epsilon)) \end{aligned} \tag{4.33}$$

and this gives us the lower bound on $E_{\tilde{\Delta}}(H_0)[H_0, \mathrm{i}A]E_{\tilde{\Delta}}(H_0)$ in the form

$$\begin{aligned} E_{\tilde{\Delta}}(H_0)[H_0, \mathrm{i}A]E_{\tilde{\Delta}}(H_0) &= E_{\tilde{\Delta}}(H_0)(-2\beta\, \partial_x^2)E_{\tilde{\Delta}}(H_0) \\ &= \sum_{n\leqslant n_0(E)} P_n\, \chi_t([E-(2n+1)\alpha-\epsilon, E-(2n+1)\alpha+\epsilon))(-2\beta\, \partial_x^2) \\ &\quad \times P_n\, \chi_t([E-(2n+1)\alpha-\epsilon, E-(2n+1)\alpha+\epsilon)) \geqslant E_{\tilde{\Delta}}(H_0)\, 2\,\delta. \end{aligned} \tag{4.34}$$

Applying the argument of [FGW] this result can be extended to H. For $I(\alpha,\delta) \not\supset \Delta \ni E$ we decompose $E_\Delta(H)$ as

$$E_\Delta(H) = E_{\tilde{\Delta}}(H_0)E_\Delta(H) + (1 - E_{\tilde{\Delta}}(H_0))E_\Delta(H)$$

and since $E_{\tilde{\Delta}}(H_0)$ commutes with $[H_0, \mathrm{i}A]$ we get

$$\begin{aligned} E_\Delta(H)\,([H_0, \mathrm{i}A] - 2\,\delta)\, E_\Delta(H) &= E_\Delta(H)E_{\tilde{\Delta}}(H_0)([H_0, \mathrm{i}A] - 2\,\delta)E_{\tilde{\Delta}}(H_0)E_\Delta(H) \\ &\quad + E_\Delta(H)([H_0, \mathrm{i}A] - 2\,\delta)(1 - E_{\tilde{\Delta}}(H_0))E_\Delta(H). \end{aligned} \tag{4.35}$$

From this one easily obtains the following inequality:

$$\begin{aligned} E_\Delta(H)\,([H_0, \mathrm{i}A] - 2\,\delta)\, E_\Delta(H) \leqslant \geqslant & E_\Delta(H)E_{\tilde{\Delta}}(H_0)([H_0, \mathrm{i}A] - 2\,\delta)E_{\tilde{\Delta}}(H_0)E_\Delta(H) \\ & - \|([H_0, \mathrm{i}A] - 2\,\delta)(1 - E_{\tilde{\Delta}}(H_0))E_\Delta(H)\| \end{aligned} \tag{4.36}$$

where the first term on the rhs is non-negative. From lemma 4.1 we know that

$$\|\beta\, \partial_x^2\, H_0^{-1}\| \leqslant \beta\, C(\omega, B) = c\, \frac{1+\alpha^2}{\alpha^2} \tag{4.37}$$

where c is a numerical constant. We can thus follow [FGW] and claim that the second term is bounded from above by

$$\begin{aligned} & 2\beta C(\omega, B)\|H_0(1 - E_{\tilde{\Delta}}(H_0))(H_0 - E)^{-1}\|\, \|(H_0 - E)E_\Delta(H)\| \\ & \quad + 2\,\delta\, \|H_0^{-1}\|\, \|H_0(1 - E_{\tilde{\Delta}}(H_0))(H_0 - E)^{-1}\|\, \|(H_0 - E)E_\Delta(H)\| \\ & \leqslant 2(\delta\alpha^{-1} + \beta C(\omega, B))(1 + E\,\epsilon^{-1})(|\Delta| + W_0) \end{aligned} \tag{4.38}$$

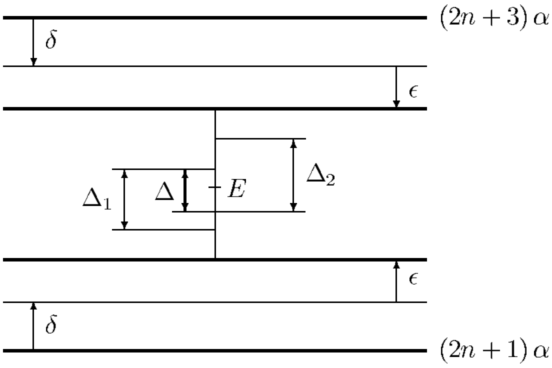

Figure 1. Energy intervals for the Mourre estimate.

so that for

$$(|\Delta| + W_0) < \frac{\delta}{2(\delta\alpha^{-1} + \beta C(\omega, B))(1 + E\epsilon^{-1})} \tag{4.39}$$

is

$$E_\Delta(H)([H_0, \mathrm{i}A] - 2\,\delta)E_\Delta(H) \geqslant -\delta$$

and hence

$$E_\Delta(H)[H_0, \mathrm{i}A]E_\Delta(H) \geqslant \delta E_\Delta(H) \tag{4.40}$$

which is what we set out to prove. □

Armed with these lemmas we are in a position to prove the Mourre estimate for H.

Lemma 4.5. *Let $E \notin I(\alpha, \delta + \epsilon)$. Assume moreover that*

$$(I) \qquad W_0 < \frac{\delta}{2(\delta\alpha^{-1} + \beta C(\omega, B))(1 + E\epsilon^{-1})}$$

and

$$(II) \qquad W_0' + B\alpha^{-2}\sqrt{c\,C(\omega, B)}\,W_0\,(E + W_0) < \delta/2. \tag{4.41}$$

Then there is an open interval $\Delta \ni E$ such that

$$E_\Delta(H)[H, \mathrm{i}A]E_\Delta(H) \geqslant \delta/2\,E_\Delta(H). \tag{4.42}$$

Proof. Consider again some open interval $\Delta_1 \ni E$, see figure 1, and a state $\psi = E_{\Delta_1}(H)\psi$. We mimick the argument used in the proof of lemma 4.2 and keeping in mind that $\|(H - E)\psi\| \leqslant |\Delta_1|\,\|\psi\|$ we get

$$\begin{aligned} |(\psi, [W, \mathrm{i}A]\psi)| &\leqslant W_0'\,\|\psi\|^2 + 2B\alpha^{-2}W_0\|\partial_x\partial_y\psi\|\,\|\psi\| \\ &\leqslant W_0'\,\|\psi\|^2 + B\alpha^{-2}\sqrt{c\,C(\omega, B)}\,W_0(E + W_0 + |\Delta_1|)\,\|\psi\|^2 \end{aligned} \tag{4.43}$$

where we have used the fact that $2\|\partial_x\partial_y H_0^{-1}\| \leqslant \sqrt{c\,C(\omega, B)}$, see lemma 4.1.

By letting $|\Delta_1| \to 0$ we get from (4.41) the upper bound on the contribution from $W(x, y)$:

$$|(\psi, [W, \mathrm{i}A]\psi)| < \delta/2\,\|\psi\|^2. \tag{4.44}$$

On the other hand by lemma 4.4 for W_0 sufficiently small there is $\Delta_2 \ni E$ such that

$$(\psi, [H_0, \mathrm{i}A]\psi) \geqslant \delta\,\|\psi\|^2 \tag{4.45}$$

for $\psi = E_{\Delta_2}(H)\psi$.

To complete the proof it suffices to take $\Delta = \Delta_1 \cap \Delta_2$. □

Note that once the condition (4.41) holds for some $\tilde{E}$, it holds also for all $E \leqslant \tilde{E}$. This leads us to the following definition:

$$\Delta(E, \alpha, \delta + \epsilon) := \{\lambda |\, \lambda \leqslant E,\ \lambda \notin I(\alpha, \delta + \epsilon)\}. \tag{4.46}$$

Now we are ready to state our main result.

Theorem 4.3. *Assume $W_0 = \|W\|_\infty < \alpha$, $W_0' = \|x\partial_x W\|_\infty < \infty$ and that the assumptions of lemma (4.5) are satisfied for some ϵ and $E \notin I(\alpha, \delta + \epsilon)$. Then:*

(1) H has no eigenvalues in the set $\Delta(E, \alpha, \delta + \epsilon)$,
(2) if in addition $W \in C^2(\mathbb{R}^2)$ and

$$\|\partial_x^2 W\|_\infty < \infty,\ \|\partial_y^2 W\|_\infty < \infty,\ \|\partial_x \partial_y W\|_\infty < \infty,\ \|x^2 \partial_x^2 W\|_\infty < \infty$$

then the spectrum of H is in $\Delta(E, \alpha, \delta + \epsilon)$ purely absolutely continuous.

Proof. Application of the Virial and Mourre theorems respectively, and lemmas 4.2, 4.3 and 4.5. □

Remark. Theorem 4.3 does not exclude the possibility that the spectrum of H is empty in the considered interval. However, it follows from the standard perturbative argument that since the spectrum of $H_0 = H - W$ includes the whole interval $[\alpha, \infty)$ this cannot happen for W_0 small enough.

Let us now consider the following scaling:

$$E = E_0\,\alpha \qquad \delta = \delta_0\,\alpha \qquad \epsilon = \epsilon_0\,\alpha$$

where E_0, δ_0, ϵ_0 are fixed. From (I) we then get

$$W_0 < \frac{\delta_0 \epsilon_0 \alpha}{2\left(\delta_0 + c\frac{1+\alpha^2}{\alpha^2}\right)(\epsilon_0 + E_0)} \to \infty \qquad \text{as} \quad \omega \to \infty \tag{4.47}$$

and similarly from (II)

$$W_0' < \alpha\,\delta_0/2 - cB\alpha^{-2}\sqrt{\frac{1+\alpha^2}{\alpha^2}}\, W_0(E_0\,\alpha + W_0) \to \infty \qquad \text{as} \quad \omega \to \infty. \tag{4.48}$$

In other words, for ω sufficiently large there is some interval in between the modified Landau levels, in which the transport survives whenever W_0, $W_0' < \infty$. We thus have

Corollary 4.1. *Let E_0, δ_0, ϵ_0 be fixed and assume that both W_0 and W_0' are finite. Then the statements of theorem 4.3 hold in the set $\Delta(\alpha E_0, \alpha, \alpha(\delta_0 + \epsilon_0))$ provided ω is large enough.*

On the other hand, in the high-energy limit the behaviour of the bound (I) is as E^{-1}. Accordingly, theorem 4.3 proves the absence of eigenvalues respectively absolute continuity only in a finite number of intervals. In this sense our result is comparable with those of [FGW, BP], where the upper bound on the size of perturbation is also $\mathcal{O}(E^{-1})$ as $E \to \infty$. For comparison we note that the same bound on $\|W\|_\infty$ obtained in [MMP] is decreasing with energy as E^{-4}.

4.3. The positivity of $[H_0, \mathrm{i}A]$: a more general approach

As we have seen above, the condition $W_0' < \infty$ which does not allow us to consider non-localized perturbations, e.g. random, comes from the fact that our conjugate operator includes the dilation generator xp_x. Let us now show that, for A being a quadratic function of (x, y, p_x, p_y), the presence of this term is necessary if one requires $[H_0, \mathrm{i}A]$ to be definite positive.

We take A in the form

$$A = \sum_{j,k} \alpha_{j,k} \partial_{x_j} \partial_{x_k} + \mathrm{i} \sum_{j,k} \beta_{j,k} (x_k \partial_{x_j} + \partial_{x_j} x_k) + \sum_{j,k} \gamma_{j,k} x_j x_k + \mathrm{i} \sum_j \delta_j \partial_{x_j} + \sum_j \epsilon_j x_j \tag{4.49}$$

where $j, k = 1, 2$. Assume that the 'bad' term is absent, i.e. $\beta_{1,1} = 0$. The straightforward computation then gives

$$\begin{aligned}
[H_0, \mathrm{i}A] = {} & 4B\alpha_{1,2}\, p_1^2 + 2(B\alpha_{2,2} - \beta_{1,2} - \beta_{2,1})\, p_1 p_2 - 4\beta_{2,2}\, p_2^2 + 4\gamma_{1,2}\, x_1 p_2 \\
& + (2\gamma_{1,1} + B\beta_{2,1})(x_1 p_1 + p_1 x_1) + 4(\alpha^2 \alpha_{1,2} + \gamma_{1,2} - B\beta_{2,2})\, x_2 p_1 \\
& + 2(2\alpha^2 \alpha_{2,2} + 2\gamma_{2,2} - B\beta_{2,1})(x_2 p_2 + p_2 x_2) \\
& + 4(\alpha^2 \beta_{2,1} + B\gamma_{1,1})\, x_1 x_2 + 4(\alpha^2 \beta_{2,2} + B\gamma_{1,2})\, x_2^2 + \mathrm{i}(\epsilon_1 p_1 + \epsilon_2 p_2) \\
& + 2\delta_2 \alpha^2\, x_2 - 2\mathrm{i}(\gamma_{1,1} + \gamma_{2,2} + \alpha^2 \alpha_{2,2}).
\end{aligned} \tag{4.50}$$

First of all notice that since H_0 is purely quadratic, the linear terms of A produce again only linear terms in $[H_0, \mathrm{i}A]$ and we can thus leave them out without loss of generality. The central point is that, due to the translation invariance in x, the term proportional to x_1^2 is missing in $[H_0, \mathrm{i}A]$. This means that if we want $[H_0, \mathrm{i}A]$ to be definite positive, we have to make the terms with x_1 vanish:

$$\gamma_{1,2} = 0 \qquad 2\gamma_{1,1} + B\beta_{2,1} = 0 \qquad \alpha^2 \beta_{2,1} + B\gamma_{1,1} = 0. \tag{4.51}$$

But now x_2^2 and p_2^2 have necessarily opposite signs, so that we also need $\beta_{2,2}$ to be zero, which implies that x_2^2 is absent as well. Following the argument given above for x_1^2 we get

$$\alpha_{1,2} = 0 \qquad 2\alpha^2 \alpha_{2,2} + 2\gamma_{2,2} - B\beta_{2,1} = 0 \tag{4.52}$$

and we are left with

$$2(B\alpha_{2,2} - \beta_{1,2} - \beta_{2,1})\, p_1 p_2$$

which cannot be definite positive.

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Appendix

Proof of lemma 4.1. Application of a partial Fourier transform in x shows that H_0 is unitarily equivalent to

$$\hat{H}_0 = -\partial_v^2 + u^2 + 2Buv + \alpha^2 v^2 = P^2 + V(u, v) \tag{A.1}$$

where $P := -\mathrm{i}\partial_v$. We now mimick the argument used in [BEH, example 7.2.4]. First of all note that since

$$u^2 + 2Buv + \alpha^2 v^2 = (u + Bv)^2 + \omega^2 v^2$$

we can write

$$V(u, v) = (V^{1/2}(u, v))^2.$$

For $\psi \in S(\mathbb{R}^2)$

$$\begin{aligned} \|(P^2 + V)\psi\|^2 &= (\psi, (P^4 + V^2 + P^2 V + V P^2)\psi) \\ &= (\psi, (P^4 + V^2 + 2PVP + [P, [P, V]])\psi). \end{aligned} \tag{A.2}$$

Furthermore, we compute

$$[P, [P, V]] = [P, -\mathrm{i}\partial_v V] = -\partial_v^2 V = -2\alpha^2.$$

Then

$$\|(P^2 + V)\psi\|^2 = \|P^2\psi\|^2 + \|V\psi\|^2 + 2\|V^{1/2} P\psi\|^2 - 2\alpha^2\|\psi\|^2$$

so that

$$\|P^2\psi\|^2 + \|V\psi\|^2 \leqslant 2\alpha^2\|\psi\|^2 + \|(P^2 + V)\psi\|^2.$$

Since both P^2, V are closed we can follow the argument given in [BEH, example 7.2.4] and claim that

$$D(P^2 + V) = D(P^2) \cap D(V). \tag{A.3}$$

Taking $\hat{R}_0(\lambda) = (\hat{H}_0 + \lambda)^{-1}$ for some $\lambda > 0$ it then follows from closed graph theorem that both

$$P^2\hat{R}_0(\lambda) \qquad V\hat{R}_0(\lambda)$$

are bounded. More precisely, one can show that for any $\psi \in S(\mathbb{R}^2)$

$$\|P^2\hat{R}_0(\lambda)\psi\| \leqslant \sqrt{6}\,\|\psi\| \qquad \|V\hat{R}_0(\lambda)\psi\| \leqslant \sqrt{6}\,\|\psi\| \tag{A.4}$$

which proves (i). To continue we note that $V(u, v)$ can be diagonalized by an orthogonal transform T so that

$$V(u, v) = \lambda_+\hat{u}^2 + \lambda_-\hat{v}^2 \tag{A.5}$$

where $(\hat{u}, \hat{v}) = T(u, v)$ and

$$\lambda_\pm = \frac{1 + \alpha^2 \pm \sqrt{(1 + \alpha^2)^2 - 4\omega^2}}{2}.$$

Therefore we have

$$\begin{aligned} V(u, v) \geqslant \lambda_-(u^2 + v^2) &= \frac{1}{2}\,\frac{(1 + \alpha^2)^2 - (1 + \alpha^2)^2 + 4\omega^2}{1 + \alpha^2 + \sqrt{(1 + \alpha^2)^2 - 4\omega^2}}\,(u^2 + v^2) \\ &\geqslant \frac{\omega^2}{1 + \alpha^2}\,(u^2 + v^2). \end{aligned} \tag{A.6}$$

From (A.1) we know that there exists a unitary operator U such that

$$\hat{H}_0 = U H_0 U^{-1}.$$

Now taking $\varphi = U\psi$ we get

$$\|\partial_x^2\psi\| = \|u^2\varphi\| \leqslant \frac{1 + \alpha^2}{\omega^2}\|V\varphi\| \qquad \|y^2\psi\| = \|v^2\varphi\| \leqslant \frac{1 + \alpha^2}{\omega^2}\|V\varphi\| \tag{A.7}$$

and

$$\|y\partial_x\psi\| = \|uv\varphi\| \leqslant \frac{1}{2}\,\frac{1 + \alpha^2}{\omega^2}\|V\varphi\| \tag{A.8}$$

which gives us (ii). Finally,

$$\|\partial_x\partial_y\psi\|^2 = (uP\varphi, uP\varphi) \leqslant \|P^2\varphi\|\,\|u^2\varphi\| \leqslant c^2\,\frac{1 + \alpha^2}{\omega^2}\,\|\hat{H}_0\varphi\|^2. \tag{A.9}$$

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